# Mathematics in Financial Risk Management

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**Abstract**

The paper gives an overview of mathematical models and methods used in financial risk management; the main area of application is credit risk. A brief introduction explains the mathematical issues arising in the risk management of a portfolio of loans. The paper continues with a formal overview of credit risk management models and discusses axiomatic approaches to risk measurement. We close with a section on dynamic credit risk models used in the pricing of credit derivatives. Mathematical techniques used stem from probability theory, statistics, convex analysis and stochastic process theory.

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# 1 Introduction

## 1.1 Financial Risk Management

Broadly speaking, risk management can be defined as a discipline for "Living with the possibility that future events may cause adverse effects" (Kloman 1999). In the context of risk management in financial institutions such as banks or insurance companies these adverse effects usually correspond to large losses on a portfolio of assets. Specific examples include: losses on a portfolio of market-traded securities such as stocks and bonds due to falling market prices (a so-called market risk event); losses on a pool of bonds or loans, caused by the default of some issuers or borrowers (credit risk); losses on a portfolio of insurance contracts due to the occurrence of large claims (insurance- or underwriting risk). An additional risk category is operational risk, which includes losses resulting from inadequate or failed internal processes, fraud or litigation.

In financial markets, there is in general no so-called "free lunch" or, in other words, no profit without risk. This is the reason why financial institutions actively take on risks. The role of financial risk management is to measure and manage these risks. Hence risk management can be seen as a core competence of an insurance company or a bank: by using its expertise and its capital, a financial institution can take on risks and manage them by various techniques such as diversification, hedging, or repackaging risks and transferring them back to markets, etc. While risk management has thus always been an integral part of the banking and insurance business, recent years have witnessed a large increase in the use of quantitative and mathematical

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techniques. Even more, regulators and supervisory authorities nowadays even require banks to use quantitative models as part of their risk management process.

Given the random nature of future events on financial markets, the field of stochastics (probability theory, statistics and the theory of stochastic processes) obviously plays an important role in quantitative risk management. In addition, techniques from convex analysis and optimization and numerical methods are frequently being used. In fact, part of the challenge in quantitative risk management stems from the fact that techniques from several existing quantitative disciplines are drawn together. The ideal skill-set of a quantitative risk manager includes concepts and techniques from such fields as mathematical finance and stochastic process theory, statistics, actuarial mathematics, econometrics and financial economics, combined of course with non-mathematical skills such as a sound understanding of financial markets and the ability to interact with colleagues with diverse training and background.

In this paper we give an introduction to some of the mathematical aspects of financial risk management. We have chosen the problem of measuring and managing the risks associated with a portfolio of bonds or loans as vehicle for our discussion. This choice is motivated by our common research interests; moreover, quantitative credit risk models are currently a hot topic in academia and industry.

## 1.2 Risk Management for a Loan Portfolio

**The loss distribution.** Consider a portfolio of loans to $m$ different counterparties, indexed by $i \in \{1, \dots, m\}$. The standard way for measuring the risk in this portfolio is to look at the change in the portfolio-value over a fixed time horizon $T$ such as one year (current time is $t = 0$). We start with a single loan with given exposure (size) $e_i$ and maturity date (repayment date) bigger than $T$. The main risk is default risk, i.e. the risk that the borrower cannot repay the loan in full. Denote by $\tau_i > 0$ the random default time of borrower $i$ and introduce the Bernoulli random variable

$$Y_i = 1_{\{\tau_i \le T\}} := \begin{cases} 1, & \text{if } \tau_i \le T\,, \\ 0, & \text{else}\,. \end{cases} \tag{1}$$

Assume that in case of default the borrower pays the lender the amount $(1 - \delta_i)e_i$, $\delta_i \in (0, 1]$ being the proportion of the exposure which is lost in default (the so-called *relative loss given default*). Abstracting from interest-rate payments the potential loss generated by loan $i$ over the period $(0, T]$ is then given by $L_i = \delta_i e_i Y_i$. Denote by

$$\bar{p}_i := \mathbb{P}(Y_i = 1) = \mathbb{P}(\tau_i \le T) \tag{2}$$

the *default probability* of counterparty $i$; $\bar{p}_i$ is by definition the probability that loan $i$ causes a loss and plays therefore an important role in measuring the default risk of the loan.

The loss of the whole portfolio of $m$ firms is then given by $L = \sum_{i=1}^{m} e_i \delta_i Y_i$. In realistic applications $m$ can be quite large: loan portfolios of major commercial banks contain several million loans. The *portfolio loss distribution* is then determined by $F_L(l) = \mathbb{P}(L \le l)$. Note that $F_L$ depends on the multivariate distribution of the random vector $(Y_1, \dots, Y_m)$ and not just on the individual default probabilities $\bar{p}_i, 1 \le i \le m$. In order to determine $F_L$ we hence need a proper mathematical model for the joint distribution of $(Y_1, \dots, Y_m)$; this issue is taken up in Section 2.2.

Dependence between defaults can have a large impact on the form of $F_L$ and in particular on its right tail (the probability of large losses). This is illustrated in Figure 1, where we compare the loss distribution for a portfolio of 1000 firms that default independently (portfolio 1) with a more realistic portfolio of the same size where defaults are dependent (portfolio 2). In portfolio 2 defaults are weakly dependent in the sense that the correlation between default events

to risk aversion: spectral risk measures can be represented as integrals specified by appropriate risk aversion functions $\sigma$ (see Theorem 3.7).

Recall that two real valued random variables $X$ and $Y$ are said to be *comonotonic* if there exist a real valued random variable $Z$ and two non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$. A risk measure $\rho$ will be called *law-invariant* if $\rho(X)$ depends only on the distribution of $X$. Note that VaR and Expected Shortfall are law-invariant. Furthermore, it has been recently shown in Jouini et al. (2006) that law-invariant convex risk measures have the Fatou property.

**Definition 3.6 (Spectral risk measures).** A coherent risk measure $\rho$ is called a spectral risk measure if it is law-invariant and comonotonic additive, meaning that $\rho(X+Y) = \rho(X) + \rho(Y)$ for all comonotonic $X, Y \in V$.

Law invariance of a risk measure $\rho$ is an essential property for practical applications: note that a risk measure can only be estimated from empirical loss data if it is law-invariant. Two comonotonic portfolios $X, Y \in V$ provide no diversification at all when added together. It is therefore a natural requirement that $\rho(X+Y)$ should equal the sum of $\rho(X)$ and $\rho(Y)$. If a risk measure is subadditive and comonotonic additive the upper bound $\rho(X) + \rho(Y)$ placed on $\rho(X+Y)$ by subadditivity is sharp as it can be actually attained in the case of comonotonic variables.

For a proof of the following theorem we refer to Kusuoka (2001), Acerbi (2002) and Tasche (2002). Generalizations can be found in Föllmer & Schied (2004) and Weber (2004).

**Theorem 3.7 (Characterization of spectral risk measures).** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with non-atomic $\mathbb{P}$, i.e. there exists a random variable that is uniformly distributed on (0,1). Then the following three conditions are equivalent for a risk measure $\rho$.*

1. *$\rho$ is a spectral measure of risk.*
2. *$\rho$ is in the convex hull of the Expected Shortfall measures.*
3. *$\rho$ can be represented in the form*

$$\rho(X) = p \int_0^1 \mathrm{VaR}_u(X)\sigma(u)du + (1-p)\mathrm{VaR}_1(X)$$

*where $p \in [0,1]$ and $\sigma$ is a non-decreasing density on $[0,1]$, i.e. $\sigma \geq 0$ on $[0,1]$, $\int_0^1 \sigma(u)du = 1$, and $\sigma(u_1) \leq \sigma(u_2)$ for $0 \leq u_1 \leq u_2 \leq 1$.*

## 3.6 Capital Allocation

We now turn to the allocation of risk capital either to subportfolios or to business units. More formally, assume that a risk measure $\rho$ has been fixed and let $X$ be a portfolio which consists of subportfolios $X_1, \ldots, X_m$, i.e. $X = X_1 + \ldots + X_m$. The objective is to distribute the risk capital $k := \rho(X)$ of the portfolio $X$ to its subportfolios, i.e. to compute risk contributions $k_1, \ldots, k_m$ of $X_1, \ldots, X_m$ with $k = k_1 + \ldots + k_m$.

Allocation techniques for risk capital are a prerequisite for portfolio management and performance measurement. In recent years, theoretical and practical aspects of different allocation schemes have been analyzed in a number of papers; see for instance Tasche (1999, 2002), Overbeck (2000), Delbaen (2000), Denault (2001), Hallerbach (2003). An allocation scheme proposed by several authors is the *allocation by the gradient* or *Euler principle*:[2] the capital allocated to

[2]Recall Euler's well-known rule that states that if $f : S \to \mathbb{R}$ is positively homogeneous and differentiable at $x \in S \subseteq \mathbb{R}^n$, we have $f(x) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x)$.

the subportfolio $X_i$ of $X$ is the derivative of the associated risk measure $\rho$ at $X$ in the direction of $X_i$ (see (24) for a precise formalization). Tasche (1999) argues that allocation based on the Euler principle provides the right signals for performance measurement. Another justification for the Euler principle is given in Denault (2001) using cooperative game theory and the notion of "fairness". He shows that the Euler principle is the only fair allocation principle for a coherent risk measure. In the following we will review a simple axiomatization of capital allocation in Kalkbrener (2005). The main axioms are the property that the entire risk capital of a portfolio is allocated to its subportfolios and a diversification property that is closely linked to the subadditivity of the underlying risk measure. It turns out that in this framework the Euler principle is an immediate consequence of the proposed axioms.

The axiomatization is based on the assumption that the capital allocated to subportfolio $X_i$ only depends on $X_i$ and $X$ but not on the decomposition of the remainder $X - X_i = \sum_{j \neq i} X_j$ of the portfolio. Hence, a capital allocation can be considered as a function $\Lambda$ from $V \times V$ to $\mathbb{R}$. Its interpretation is, that $\Lambda(X, Y)$ represents the capital allocated to the portfolio $X$ considered as a subportfolio of portfolio $Y$.

**Definition 3.8 (Axiomatization of capital allocation).** A function $\Lambda\colon V \times V \to \mathbb{R}$ is called a capital allocation with respect to a risk measure $\rho$ if it satisfies the condition $\Lambda(X, X) = \rho(X)$ for all $X \in V$, i.e. if the capital allocated to $X$ (considered as stand-alone portfolio) is the risk capital $\rho(X)$ of $X$.
The following requirements for a capital allocation $\Lambda$ are proposed.

1. **Linearity.** For a given overall portfolio $Z$ the capital allocated to a union of subportfolios is equal to the sum of the capital amounts allocated to the individual subportfolios. In particular, the risk capital of a portfolio equals the sum of the risk capital of its subportfolios. More formally, $\Lambda$ is called linear if
$$\forall a, b \in \mathbb{R}, X, Y, Z \in V \quad \Lambda(aX + bY, Z) = a\Lambda(X, Z) + b\Lambda(Y, Z).$$
2. **Diversification.** The capital allocated to a subportfolio $X$ of a larger portfolio $Y$ never exceeds the risk capital of $X$ considered as a stand-alone portfolio: $\Lambda$ is called diversifying if
$$\forall X, Y \in V \quad \Lambda(X, Y) \leq \Lambda(X, X).$$
3. **Continuity.** A small increase in a position does only have a small effect on the risk capital allocated to that position: $\Lambda$ is called continuous at $Y \in V$ if
$$\forall X \in V \quad \lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y).$$

Risk measures and capital allocation rules are closely related. First, given a capital allocation $\Lambda$ the corresponding risk measure $\rho$ is obviously given by the values of $\Lambda$ on the diagonal, i.e. $\rho(X) = \Lambda(X, X)$. Conversely, for a positively homogeneous and subadditive risk measure $\rho$ a corresponding capital allocation $\Lambda_\rho$ can be constructed as follows: let $V^*$ be the set of real linear functionals on $V$ and for a given risk measure $\rho$ consider the following subset
$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$
It is an easy consequence of the Hahn-Banach Theorem that for a positively homogeneous and subadditive risk measure $\rho$
$$\rho(X) = \max\{h(X) \mid h \in H_\rho\} \tag{22}$$

for all $X \in V$. Hence for every $Y \in V$ there exists an $h_Y^\rho \in H_\rho$ with $h_Y^\rho(Y) = \rho(Y)$. This allows to define a capital allocation $\Lambda_\rho$ by

$$\Lambda_\rho(X, Y) := h_Y^\rho(X). \tag{23}$$

The set $H_\rho$ can be interpreted as a collection of (generalized) scenarios: the capital allocated to a subportfolio $X$ of portfolio $Y$ is simply the loss of $X$ under scenario $h_Y^\rho$.

The following theorem (Theorem 4.2 in Kalkbrener (2005)) states the equivalence between positively homogeneous, subadditive (but not necessarily monotonic) risk measures and linear, diversifying capital allocations.

**Theorem 3.9 (Existence of capital allocations).** *Let $\rho\colon V \to \mathbb{R}$.*

**a)** *If there exists a linear, diversifying capital allocation $\Lambda$ with associated risk measure $\rho$ then $\rho$ is positively homogeneous and subadditive.*

**b)** *If $\rho$ is positively homogeneous and subadditive then $\Lambda_\rho$ is a linear, diversifying capital allocation with associated risk measure $\rho$.*

If a linear, diversifying capital allocation $\Lambda$ is moreover continuous at a portfolio $Y \in V$ it is uniquely determined by the directional derivative of its associated risk measure, as the next theorem (Theorem 4.3 in Kalkbrener (2005)) shows.

**Theorem 3.10.** *Let $\rho$ be a positively homogeneous and sub-additive risk measure and $Y \in V$. Then the following three conditions are equivalent:*

**a)** *$\Lambda_\rho$ is continuous at $Y$, i.e. for all $X \in V$ $\lim_{\epsilon\to 0} \Lambda_\rho(X, Y + \epsilon X) = \Lambda_\rho(X, Y)$.*

**b)** *The directional derivative*

$$\lim_{\epsilon\to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon} \tag{24}$$

*exists for every $X \in V$.*

**c)** *There exists a* unique *$h \in H_\rho$ with $h(Y) = \rho(Y)$.*

*If these conditions are satisfied then $\Lambda_\rho(X, Y)$ equals (24) for all $X \in V$, i.e. $\Lambda_\rho$ is given by the Euler principle.*

Theorem 3.9 implies that in the general case, in particular for credit portfolios, there do not exist linear diversifying capital allocations for VaR since VaR is not subadditive. However, under regularity conditions (see, for example, Tasche (1999)), the directional derivative (24) exists for $\mathrm{VaR}_\alpha$ and equals

$$\mathrm{E}(X|Y = \mathrm{VaR}_\alpha(Y)). \tag{25}$$

The volatility (or covariance) allocation, on the other hand, is linear and diversifying, as it is derived from the risk measure Standard Deviation using (23). More precisely, let $c$ be a non-negative real number and define the risk measure $\rho_c^{Std}$ and the capital allocation $\Lambda_c^{Std}$ by

$$\rho_c^{Std}(X) := c \cdot \mathrm{Std}(X) + \mathrm{E}(X), \tag{26}$$

$$\Lambda_c^{Std}(X, Y) := \begin{cases} c \cdot \mathrm{Cov}(X, Y)/\mathrm{Std}(Y) + E(X) & \text{if } \mathrm{Std}(Y) > 0, \\ E(X) & \text{if } \mathrm{Std}(Y) = 0. \end{cases} \tag{27}$$

Then the risk measure $\rho_c^{Std}$ is translation invariant, positively homogeneous and subadditive but not monotonic for $c > 0$. $\Lambda_c^{Std}$ is a linear, diversifying capital allocation with respect to $\rho_c^{Std}$. If

$\mathrm{Std}(Y) > 0$ then $\Lambda_c^{Std}$ is continuous at $Y$ and equals the directional derivative (24) by Theorem 3.10.

Expected Shortfall ES is a coherent risk measure and therefore positively homogeneous and subadditive. Hence, application of (23) to Expected Shortfall yields a linear, diversifying capital allocation with associated risk measure ES. The scenario function $h_Y^{\mathrm{ES}}(X)$ for this risk measure is given by $\mathrm{E}_{\mathsf{Q}_Y}(X)$, where the probability measure $\mathbb{Q}_Y$ is specified in (20). In summary,

$$\Lambda_\alpha^{ES}(X,Y) := \mathrm{E}_{\mathsf{Q}_Y}(X) = \left(\int X \cdot \mathbf{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}} d\mathbb{P} + \beta_Y \int X \cdot \mathbf{1}_{\{Y=\mathrm{VaR}_\alpha(Y)\}} d\mathbb{P}\right) / (1-\alpha)$$

is a linear, diversifying capital allocation with respect to $\mathrm{ES}_\alpha$. If

$$\mathbb{P}(Y > \mathrm{VaR}_\alpha(Y)) = 1-\alpha \quad \text{or} \quad \mathbb{P}(Y \geq \mathrm{VaR}_\alpha(Y)) = 1-\alpha \tag{28}$$

then $\Lambda_\alpha^{ES}$ is continuous at $Y$ and equals the directional derivative (24). In particular, (28) holds if $\mathbb{P}(Y = \mathrm{VaR}_\alpha(Y)) = 0$; in that case $\Lambda_\alpha^{ES}(X,Y)$ takes the particularly intuitive form

$$\Lambda_\alpha^{ES}(X,Y) = E\left(X \mid Y > \mathrm{VaR}_\alpha(Y)\right).$$

The extension to spectral risk measures can be found in Overbeck (2004).

### 3.7 Case study: capital allocation in an investment banking portfolio

We will now analyze the practical consequences of different allocation schemes when applied to a realistic credit portfolio. The case study is based on a sample investment banking portfolio consisting of $m = 25000$ loans with an inhomogeneous exposure and default probability distribution. The average exposure size is 0.004% of the total exposure and the standard deviation of the exposure size is 0.026%. The portfolio expected loss is 0.72% and the unexpected loss, i.e. the standard deviation, is 0.87%. Default probabilities $\bar{p}_1, \ldots, \bar{p}_m$ of all companies are obtained from Deutsche Bank's rating system and vary between 0.02% and 27%. Default correlations are specified by a Bernoulli mixture model: for company $i$, the conditional default $p_i$ has the form

$$p_i(\quad) := \Phi\left(\frac{\Phi^{-1}(\bar{p}_i) - \sqrt{R_i}\sum_{j=1}^{96} \alpha_{ij}\psi_j}{\sqrt{1-R_i}}\right). \tag{29}$$

where the 96 systematic factors $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_{96})$ follow a multi-dimensional normal distribution and represent different countries and industries; see (9) and (13).

The portfolio loss distribution $L$ specified by this model does not have an analytic form. Monte Carlo simulation is therefore used for the calculation and allocation of risk capital. For this class of models, however, the Monte Carlo estimation of tail-focused risk measures like Value-at-Risk or Expected Shortfall is a demanding computational problem due to high statistical fluctuations. This stability problem is even more pronounced for Expected Shortfall contributions of individual transactions. Importance sampling is a variance reduction technique that has been successfully applied in credit portfolio models of this type. We refer to Glasserman &Li (2005), Kalkbrener et al. (2004) and Egloff et al. (2005) for details.

For the test portfolio we have calculated the risk measures $\mathrm{VaR}_{0.9998}(L)$, $\mathrm{ES}_{0.999}(L)$ and $\mathrm{ES}_{0.99}(L)$. The $\mathrm{VaR}_{0.9998}(L)$ is the risk measure used at Deutsche Bank for calculating Economic Capital, i.e. the capital requirement for absorbing unexpected losses over a one-year period with a high degree of certainty. The confidence level of 99.98% is derived from Deutsche Bank's target rating of AA+, which is associated with an annual default rate of 0.02%. The $\mathrm{ES}_{0.999}(L)$ has been chosen since it leads to a comparable amount of risk capital, while being based on a coherent risk measure. The $\mathrm{ES}_{0.99}(L)$ was calculated to study the impact of the confidence level $\alpha$ on the

rate is not a risk-free rate since by definition it is the rate at which large internationally operating banks lend money to other large internationally operating banks. There is a very small default risk involved and consequently the Libor rate is slightly above the treasury rate. Since it is readily available it is convenient to take the Libor rate as the base rate. The corresponding rate for a contract which has a nonnegligible probability to default is the *defaultable forward Libor rate* $\overline{L}(t,T_k)$. Both rates are related by the equation

$$\overline{L}(t,T_k) = L(t,T_k) + S(t,T_k) \tag{31}$$

where $S(t,T_k)$ is the (positive) spread. Since $S(t,T_k)$ turns out not to be the quantity which will show up in valuation formulae for credit derivatives we will model instead the *forward default intensities* $H(t,T_k)$ given by

$$H(t,T_k) = \frac{S(t,T_k)}{1+\delta_k L(t,T_k)}. \tag{32}$$

The term $\delta_k L(t,T_k)$ is small compared to 1, therefore, numerically $H(t,T_k)$ and $S(t,T_k)$ are quite close.

We start by specifying the dynamics of the most distant Libor rate by setting

$$L(t,T_{n-1}) = L(0,T_{n-1}) \exp\left(\int_0^t b^L(s,T_{n-1})\,ds + \int_0^t \lambda(s,T_{n-1})\,dL_s^{T^*}\right). \tag{33}$$

The fact that $L(\cdot,T_{n-1})$ is modeled as an exponential will guarantee its positivity. $\lambda(\cdot,T_{n-1})$ is a deterministic volatility structure and $L^{T^*} = (L_t^{T^*})$ is a time-inhomogeneous Lévy process which without loss of generality has the simple canonical representation

$$L_t^{T^*} = \int_0^t \sqrt{c_s}\,dW_s^{T^*} + \int_0^t \int_{\mathsf{R}} x(\mu - \nu^{T^*})(ds,dx). \tag{34}$$

The first term is a stochastic integral with respect to a standard Brownian motion $W^{T^*}$ and represents the continuous Gaussian part, whereas the second integral, which is an integral with respect to the compensated random measure of jumps of $L^{T^*}$, is a purely discontinuous process. The drift term $b^L(\cdot,T_{n-1})$ will be chosen in such a way that $L(\cdot,T_{n-1})$ becomes a martingale under the *terminal forward measure* $\mathbb{P}_{T^*}$.[4]

Via a backward induction for each tenor time point $T_k$, *forward measures* $\mathbb{P}_{T_k}$ are derived. Although one could define each forward martingale measure $\mathbb{P}_{T_k}$ by giving explicitly its density relative to the spot martingale measure $\mathbb{P}$ – this is the usual martingale measure known from stock price models – the latter is not used in the context of Libor models. One starts with a probability measure $\mathbb{P}_{T^*}$ which is interpreted as the terminal forward measure and proceeds backwards in time by introducing successively the forward measures $\mathbb{P}_{T_k}$ via Radon–Nikodym derivatives

$$\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} = \frac{1+\delta_k L(T_k,T_k)}{1+\delta_k L(0,T_k)}.$$

Then, for each tenor time point $T_k$, under $\mathbb{P}_{T_{k+1}}$ the Libor rate $L(t,T_k)$ can be given in the following uniform form

$$L(t,T_k) = L(0,T_k) \exp\left(\int_0^t b^L(s,T_k)\,ds + \int_0^t \lambda(s,T_k)\,dL_s^{T_{k+1}}\right) \tag{35}$$

where also the driving processes $L^{T_{k+1}} = (L_t^{T_{k+1}})$ have to be derived from $L^{T^*}$ during the backward induction. To implement this model one uses only mildly time-inhomogeneous Lévy processes,

[4] $P_{T^*}$ is the martingale measure corresponding to the numeraire $B(t,T^*)$, i.e. security prices expressed in units of $B(t,T^*)$ are $P_{T^*}$-martingales.

namely piecewise (time-homogeneous) Lévy processes. Typically three Lévy parameter sets – one for short, one for intermediate, and one for long maturities – are sufficient to calibrate the model to a volatility surface given by prices of interest rate derivatives such as caps, floors and swaptions. For some calibration results see Eberlein & Koval (2006), where the Lévy Libor model has been extended to a multicurrency setting.

The dynamics of the forward default intensities $H(\cdot, T_k)$ cannot be specified directly since it depends on the specification of the random time point at which a defaultable loan or bond actually defaults. There is a standard way to construct a random time for the default event. Let $\Gamma = (\Gamma_t)$ be a hazard process, that is an adapted, right-continuous, increasing process starting at 0 with $\lim_{t\to\infty} \Gamma_t = \infty$. Let $\eta$ be a uniformly distributed random variable on the interval $[0,1]$, independent of the process $(\Gamma_t)_{t\geq 0}$, possibly defined on an extension of the underlying probability space. Then

$$\tau = \inf\{t > 0 \mid e^{-\Gamma_t} \leq \eta\} \tag{36}$$

defines a stopping time with respect to the 'right' filtration which can be used to indicate default. By choosing the hazard process $\Gamma$ appropriately – only its values at the tenor time points $T_k$ matter – one can now model the forward default intensities $H(t, T_k)$ in such a way that the dynamics is described in the same simple form (35) as given for the Libor rates, namely

$$\begin{aligned} H(t,T_k) \quad &= \quad H(0,T_k) \exp\Bigg( \int_0^t b^H(s,T_k)\, ds + \int_0^t \sqrt{c_s}\gamma(s,T_k)\, dW_s^{T_{k+1}} \\ &\qquad + \int_0^t \int_{\mathbb{R}} \gamma(s,T_k) x(\mu - \nu^{T_{k+1}})(ds,dx) \Bigg). \end{aligned} \tag{37}$$

Again this is done by a backward induction along the tenor time points and as in (35) the specific form as an exponential guarantees that the forward default intensities and thus the spreads $S(t, T_k)$ are positive.

Based on this joint model for interest and default rates we can now price defaultable instruments and credit derivatives. Let us start with a *defaultable coupon bond* with $n$ coupons of a fixed amount $c$ that are promised to be paid at the dates $T_1, \ldots, T_n$. In case default happens during the life time of the bond usually not everything is lost. There is a positive recovery. To incorporate this fact in the model, suitable recovery rules have to be fixed. The most appropriate scheme is the *recovery of par rule*. The assumption is then that if a coupon bond defaults in the time interval $(T_k, T_{k+1}]$, the recovery is given by a recovery rate $\pi \in [0,1)$ times the sum of the notional amount, which we set equal to 1, and the interest accrued over the period $(T_k, T_{k+1}]$. The resulting amount is paid at time $T_{k+1}$. The promised interest payments for subsequent periods are lost.

**Theorem 4.1 (Pricing of defaultable coupon bonds).** *Under the recovery of par rule the arbitrage-free price at time $T_0 = 0$ of a defaultable bond with $n$ coupons of amount $c$ is*

$$B(0,c,n) = \overline{B}(0,T_n) + \sum_{k=0}^{n-1} \overline{B}(0,T_{k+1}) \left( c + \pi(1+c)\delta_k E_{\overline{\mathbb{P}}_{T_{k+1}}} [H(T_k,T_k)] \right), \tag{38}$$

*where $\overline{B}(0, T_k)$ are the pre-default prices of defaultable zero-coupon bonds with maturities $T_k$, which are known at time 0.*

Note that the only random variables in this pricing formula are the forward default intensities. This is the reason why we aimed at describing the dynamics of $H(\cdot, T_k)$ in a relatively simple form. The expectations are taken with respect to the (restricted) defaultable forward measures $\overline{\mathbb{P}}_{T_{k+1}}$ for the dates $T_k$. These are the appropriate martingale measures in the defaultable world.

Their Radon–Nikodym densities with respect to the (default-free) forward measures $\mathbb{P}_{T_k}$ are given by

$$\frac{d\overline{\mathbb{P}}_{T_k}}{d\mathbb{P}_{T_k}} = \frac{B(0,T_k)}{\overline{B}(0,T_k)} e^{-\Gamma_{T_k}} = \frac{B(0,T_k)}{\overline{B}(0,T_k)} \prod_{i=0}^{k-1} \frac{1}{1+\delta_i H(T_i,T_i)}. \tag{39}$$

Recall that $B(0,T_k)$ denotes the time-0 price of a default-free zero-coupon bond with maturity $T_k$. A formula similar to (38) can be obtained to price a *defaultable floating coupon bond* that pays an interest rate composed of the default-free Libor rate plus a constant spread $x$. Let us mention here that the change of measure technique is a key tool in interest rate and credit theory to obtain valuation formulae which are as simple as possible.

The most popular and heavily traded credit derivatives are *credit default swaps*. They can be used to insure defaultable financial instruments against default. In a credit default swap the protection buyer $A$ pays periodically a fixed fee to the protection seller $B$ until a prespecified credit event occurs or the final time point of the contract is reached. The credit event can be the default of a reference bond issued by a party $C$. The protection seller in turn will make a payment that covers the losses of $A$ in case the credit event happens. Of course the credit event as well as the default payment have to be clearly specified. Let us consider a standard default swap with the maturity $T_n$ where the credit event is defined to be the default of a certain fixed-coupon bond. According to the recovery scheme explained above, the default payment $A$ will receive at time $T_{k+1}$ if default happend in the period $(T_k, T_{k+1}]$ is $1-\pi(1+c)$. The periodic fee $s$, the so-called *default swap rate*, is now determined in such a way that the initial value of the contract is zero. The time-0 value of the periodic fee payments is $s\left(\sum_{k=1}^{n} \overline{B}(0,T_{k-1})\right)$ since each fee payment of size $s$ which has to be made at time $T_{k-1}$ has to be discounted by the corresponding discount factor $\overline{B}(0,T_{k-1})$. Following the standard pricing principle for a contingent claim, some nontrivial analysis shows that the initial value of the payment $A$ will receive in case of default is

$$\sum_{k=1}^{n} (1-\pi(1+c))\overline{B}(0,T_k)\delta_{k-1} E_{\overline{\mathsf{P}}_{T_k}}[H(T_{k-1},T_{k-1})]. \tag{40}$$

Equating these two sums one gets the default swap rate

$$s = \frac{1-\pi(1+c)}{\sum\limits_{k=1}^{n} \overline{B}(0,T_{k-1})} \sum_{k=1}^{n} \left(\overline{B}(0,T_k)\delta_{k-1} E_{\overline{\mathsf{P}}_{T_k}}[H(T_{k-1},T_{k-1})]\right). \tag{41}$$

The formula shows that again expectations of forward default intensities have to be evaluated under the corresponding defaultable forward measures. Another important class of credit derivatives which can be priced in this model framework are *credit default swaptions*. The holder of such an option has the right to enter a credit default swap at some prespecified time and swap rate. Credit default swaptions are typically *extension options* which are often imbedded in a credit default swap.

There is a very liquid market for credit default swaps. Therefore the current swap rates usually do not have to be determined by formula (41). Instead, credit default swaps are used as calibration instruments for the term structure of forward default intensities. In other words, given the currently quoted swap rates, (41) is used to extract the model parameters and then the so calibrated model can be used to price less liquid instruments for example in the OTC-market. Other derivatives which can be priced in this modelling framework are *total rate of return swaps*, *asset swaps*, *options on defaultable bonds*, and *credit spread options*.

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